# On all-regime, high-order and well-balanced Lagrange-Projection type schemes for the shallow water equations

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# Introduction

- Construction of Finite Volume (FV) and Discontinuous Galerkin (DG) schemes for SWE
- Theory based on a Lagrange-Projection type scheme developed for Euler Equations[1]
- Low Froude number : fast acoustic waves vs. slow material transport waves
- Acoustic - Transport operators decomposition (Lagrange - Projection like) :
  $\longrightarrow$ Impliciting fast phenomenons : less restrictive CFL condition
  $\longrightarrow$ Expliciting slow phenomenons : reasonable precision
- Linearisation of the equations with a Relaxation method

[1] Christophe Chalons, Matthieu Girardin, and Samuel Kokh. "An all-regime Lagrange-Projection like scheme for the gas dynamics equations on unstructured meshes". In: *to appear in Communications in Computational Physics* (2016).

# Contents

# Shallow Water Equations

## Euler System in 1D

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0, \\ \partial_t(\rho E) + \partial_x((\rho E + p)u) = 0. \end{cases}$$

## Shallow Water Sytem in 1D

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\frac{h^2}{2}\right) = -gh\partial_x z. \end{cases}$$

$\longrightarrow$ Two similar systems

$\longrightarrow$ Non-conservative source term in SWE

# Operators splitting

"Acoustic" / "Transport" decomposition

$$\begin{cases} \partial_t h + \qquad h\,\partial_x u + \qquad u\,\partial_x h = 0, \\ \partial_t(hu) + \quad hu\,\partial_x u + \partial_x\left(g\dfrac{h^2}{2}\right) + \quad u\,\partial_x(hu) = -gh\partial_x z. \end{cases}$$

# Operators splitting

## "Acoustic" / "Transport" decomposition

$$\underset{t^n \to t^{n+1-}}{\textit{Acoustic}} \begin{cases} \partial_t h + & h\,\partial_x u = 0, \\ \partial_t (hu) + & hu\,\partial_x u + \partial_x \left( g \dfrac{h^2}{2} \right) = -gh\partial_x z, \end{cases}$$

$$\underset{t^{n+1-} \to t^{n+1}}{\textit{Transport}} \begin{cases} \partial_t h + & u\,\partial_x h = 0, \\ \partial_t (hu) + & u\,\partial_x (hu) = 0. \end{cases}$$

# Relaxation Method

- Change of variable : $h \longrightarrow \tau = {}^1/_h$
- Approximation of $\tau(\cdot, t)\, \partial_x X$ by $\tau(\cdot, t^n)\partial_x X = \partial_m X$
- Variable $\pi$ : linearisation of the pressure $\frac{g}{2\tau^2}$

## Acoustic System

$$\begin{cases} \partial_t h + \quad h\,\partial_x u = 0, \\ \partial_t (hu) + \quad hu\,\partial_x u + \partial_x \left( g\,\dfrac{h^2}{2} \right) = -gh\partial_x z, \end{cases}$$

# Relaxation Method

- Change of variable : $h \longrightarrow \tau = {}^{1}/_{h}$
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## Acoustic System

$$\begin{cases} \partial_t \tau - \tau\,\partial_x u = 0, \\ \partial_t u + \tau\,\partial_x \left(\frac{g}{2\tau^2}\right) = -g\partial_x z, \\ \partial_t z = 0. \end{cases}$$

# Relaxation Method

- Change of variable : $h \longrightarrow \tau = {}^1/_h$
- Approximation of $\tau(\cdot, t)\,\partial_x X$ by $\tau(\cdot, t^n)\partial_x X = \partial_m X$
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## Acoustic System

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \left( \dfrac{g}{2\tau^2} \right) = -\dfrac{g}{\tau} \partial_m z, \\ \partial_t z = 0. \end{cases}$$

# Relaxation Method

- Change of variable : $h \longrightarrow \tau = {}^{1}/_{h}$
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- Variable $\pi$ : linearisation of the pressure $\frac{g}{2\tau^2}$

## Relaxed Acoustic System

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \pi = -\dfrac{g}{\tau}\partial_m z, \\ \partial_t \pi + a^2 \partial_m u = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \\ \partial_t z = 0. \end{cases}$$

# Relaxation Method

- Change of variable : $h \longrightarrow \tau = {}^1/_h$
- Approximation of $\tau(\cdot, t)\, \partial_x X$ by $\tau(\cdot, t^n)\partial_x X = \partial_m X$
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$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \pi = -\dfrac{g}{\tau}\partial_m z, \\ \partial_t \pi + a^2 \partial_m u = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \\ \partial_t z = 0. \end{cases}$$

**Prop :** Viscous approximation of the Acoustic system under the sub-characteristic condition : $a > \max(hc) = \max(\frac{1}{\tau}\sqrt{\frac{g}{\tau}})$.

# Relaxation Method

Operators splitting :

- Instantaneous relaxation step
- Homogeneous relaxed Acoustic system

## Relaxed Acoustic System

$$\begin{cases} \partial_t \tau = 0, \\ \partial_t u = 0, \\ \partial_t \pi = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \\ \partial_t z = 0, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \pi + \dfrac{g}{\tau}\partial_m z = 0, \\ \partial_t \pi + a^2 \partial_m u = 0, \\ \partial_t z = 0. \end{cases}$$

# Contents

# Approximate Riemann solver

## Homogeneous relaxed Acoustic system

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \pi + \dfrac{g}{\tau} \partial_m z = 0, \\ \partial_t \pi + a^2 \partial_m u = 0, \\ \partial_t z = 0. \end{cases}$$

⟶ Velocities : $\{-a, 0, a\}$
⟶ Hyperbolic system
⟶ 4 linearly degenerated characteristic fields

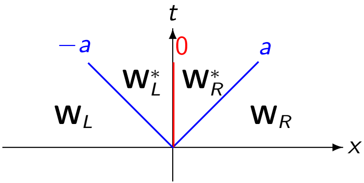
# Approximate Riemann solver

## Homogeneous relaxed Acoustic system

$$\partial_t \mathbf{W} + A \cdot \partial_m \mathbf{W} = \mathbf{0}, \text{ with } \mathbf{W} = \begin{pmatrix} \tau \\ u \\ \pi \\ z \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & g/\tau \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\longrightarrow$ Velocities : $\{-a, 0, a\}$
$\longrightarrow$ Hyperbolic system
$\longrightarrow$ 4 linearly degenerated characteristic fields

# Approximate Riemann solver



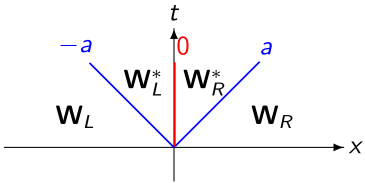

- $\mathbf{W}_R^*$, $\mathbf{W}_L^*$ : continuity of Riemann invariants across the discontinuities
- 7 Riemann invariants :

$$\begin{cases} I_\varepsilon^1 = u + \varepsilon a \tau, \\ I_\varepsilon^2 = \pi + a^2 \tau, \quad \text{associated to } \varepsilon a, \quad \text{with } \varepsilon \in \{-1, 1\}, \\ I_\varepsilon^3 = z, \end{cases}$$

and $\quad I_0^1 = u, \;$ associated to 0.

$\longrightarrow$ Not enough to find $\mathbf{W}_R^*$ and $\mathbf{W}_L^*$

# Approximate Riemann solver

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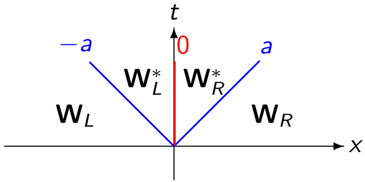
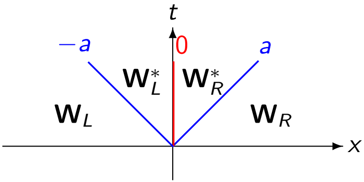
# Approximate Riemann solver




- $\mathbf{W}_{RP}$ consistent in the integral sense with SWE
- Degeneration of $\mathbf{W}_{RP}$ to the classical solution if $z = \text{cst}$
- Preservation of the "lake at rest" condition : $u_L = u_R = 0$ and $h_L + z_L = h_R + z_R$

$\longrightarrow$ A new Riemann invariant : $I_0^2 = \pi + g\frac{h_L + h_R}{2}z$ associated to 0

# Approximate Riemann solver

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$\longrightarrow$ A new Riemann invariant : $I_0^2 = \pi + g\frac{h_L+h_R}{2}z$ associated to 0

# Explicit Acoustic step

## Instantanuous relaxation step

$$\begin{cases} \partial_t \tau = \partial_t u = \partial_t z = 0, \\ \partial_t \pi = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \end{cases}$$

# Explicit Acoustic step

## Instantanuous relaxation step

$$\begin{cases} \partial_t \tau = \partial_t u = \partial_t z = 0, \\ \partial_t \pi = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \end{cases} \qquad \Rightarrow \qquad \pi_j^n := \frac{g}{2\left(\tau_j^n\right)^2} = g\frac{\left(h_j^n\right)^2}{2}.$$

# Explicit Acoustic step

## Instantanuous relaxation step

$$\begin{cases} \partial_t \tau = \partial_t u = \partial_t z = 0, \\ \partial_t \pi = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \end{cases} \qquad \Rightarrow \qquad \pi_j^n := \frac{g}{2\left(\tau_j^n\right)^2} = g\frac{\left(h_j^n\right)^2}{2}.$$

## Homogeneous relaxed Acoustic system

$$\begin{cases} \partial_t \tau - \partial_m u = 0, \\ \partial_t u + \partial_m \pi + \dfrac{g}{\tau}\partial_m z = 0, \\ \partial_t \pi + a^2 \partial_m u = 0, \\ \partial_t z = 0. \end{cases}$$

# Explicit Acoustic step

## Instantanuous relaxation step

$$\begin{cases} \partial_t \tau = \partial_t u = \partial_t z = 0, \\ \partial_t \pi = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \end{cases} \qquad \Rightarrow \qquad \pi_j^n := \frac{g}{2\left(\tau_j^n\right)^2} = g\frac{\left(h_j^n\right)^2}{2}.$$

## Homogeneous relaxed Acoustic system

Mean value of $X = \tau, u, \pi$ on cell $\kappa_j$ :

$$X_j^{n+1^-} = \frac{a\Delta t}{\Delta m_j} X_{j-1/2,R}^{*,n} + \left(1 - \frac{2a\Delta t}{\Delta m_j}\right) X_j^n + \frac{a\Delta t}{\Delta m_j} X_{j+1/2,L}^{*,n}.$$

# Explicit Acoustic step

## Instantanuous relaxation step

$$\begin{cases} \partial_t \tau = \partial_t u = \partial_t z = 0, \\ \partial_t \pi = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \end{cases} \quad \Rightarrow \quad \pi_j^n := \frac{g}{2\left(\tau_j^n\right)^2} = g\frac{\left(h_j^n\right)^2}{2}.$$

## Homogeneous relaxed Acoustic system

$$\begin{cases} \tau_j^{n+1^-} = \tau_j^n + \dfrac{\Delta t}{\Delta m_j}\left(u_{j+1/2,L}^{*,n} - u_{j-1/2,R}^{*,n}\right), \\ u_j^{n+1^-} = u_j^n - \dfrac{\Delta t}{\Delta m_j}\left(\pi_{j+1/2,L}^{*,n} - \pi_{j-1/2,R}^{*,n}\right), \\ \pi_j^{n+1^-} = \pi_j^n - a^2\dfrac{\Delta t}{\Delta m_j}\left(u_{j+1/2,L}^{*,n} - u_{j-1/2,R}^{*,n}\right). \end{cases}$$

# Explicit Acoustic step

## Instantanuous relaxation step

$$\begin{cases} \partial_t \tau = \partial_t u = \partial_t z = 0, \\ \partial_t \pi = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \end{cases} \qquad \Rightarrow \qquad \pi_j^n := \frac{g}{2\left(\tau_j^n\right)^2} = g\frac{\left(h_j^n\right)^2}{2}.$$

## Homogeneous relaxed Acoustic system

$$\begin{cases} \tau_j^{n+1^-} = \tau_j^n + \dfrac{\Delta t}{\Delta x}\tau_j^n\left(u_{j+1/2}^{*,n} - u_{j-1/2}^{*,n}\right) = L_j^n \tau_j^n, \\ u_j^{n+1^-} = u_j^n - \dfrac{\Delta t}{\Delta x}\tau_j^n\left(\pi_{j+1/2}^{*,n} - \pi_{j-1/2}^{*,n}\right) - \Delta t \left\{\dfrac{g}{\tau}\partial_m z\right\}_j^n, \\ \pi_j^{n+1^-} = \pi_j^n - a^2\dfrac{\Delta t}{\Delta x}\tau_j^n\left(u_{j+1/2}^{*,n} - u_{j-1/2}^{*,n}\right). \end{cases}$$

# Implicit Acoustic step

## Instantanuous relaxation step

$$\begin{cases} \partial_t \tau = \partial_t u = \partial_t z = 0, \\ \partial_t \pi = -\dfrac{\pi - \frac{g}{2\tau^2}}{\varepsilon}, \end{cases} \Rightarrow \qquad \pi_j^n := \frac{g}{2\left(\tau_j^n\right)^2} = g\frac{\left(h_j^n\right)^2}{2}.$$

## Homogeneous relaxed Acoustic system

$$\begin{cases} \tau_j^{n+1^-} = \tau_j^n + \dfrac{\Delta t}{\Delta x}\tau_j^n\left(u_{j+1/2}^{*,n+1^-} - u_{j-1/2}^{*,n+1^-}\right) = L_j^{n+1^-}\tau_j^n, \\ u_j^{n+1^-} = u_j^n - \dfrac{\Delta t}{\Delta x}\tau_j^n\left(\pi_{j+1/2}^{*,n+1^-} - \pi_{j-1/2}^{*,n+1^-}\right) - \Delta t\left\{\dfrac{g}{\tau}\partial_m z\right\}_j^n, \\ \pi_j^{n+1^-} = \pi_j^n - a^2\dfrac{\Delta t}{\Delta x}\tau_j^n\left(u_{j+1/2}^{*,n+1^-} - u_{j-1/2}^{*,n+1^-}\right). \end{cases}$$

# Transport step

## Transport step

$$\begin{cases} \partial_t h + u\,\partial_x h = 0, \\ \partial_t (hu) + u\,\partial_x (hu) = 0. \end{cases}$$

# Transport step

## Transport step

Mean value of $X = h, hu$ on cell $\left[x_{j-1/2}, x_{j+1/2}\right]$ :

$$X_j^{n+1} = \left(u^*_{j-1/2}\right)_+ \frac{\Delta t}{\Delta x} X_{j-1}^{n+1^-} + \left(1 - \frac{\Delta t}{\Delta x}\left[\left(u^*_{j-1/2}\right)_+ - \left(u^*_{j+1/2}\right)_-\right]\right) X_j^{n+1^-} - \left(u^*_{j+1/2}\right)_- \frac{\Delta t}{\Delta x} X_{j+1}^{n+1^-}.$$

$$u^*_{j+1/2} = \begin{cases} u^{*,n}_{j+1/2} & \text{(explicit Acoustic step)}, \\ u^{*,n+1^-}_{j+1/2} & \text{(implicit Acoustic step)}. \end{cases}$$

# Contents

# Notations

- Based on work from Florent Renac[2] at ONERA
- Lagrange polynomials on Gauss-Lobatto quadrature:

$$\rho(x) = \sum_{i=0}^{p} \rho_{i,j}\phi_{i,j}(x), \qquad \forall x \in \left[x_{j-1/2}, x_{j+1/2}\right]$$

with $\phi_{i,j}(x) = \ell_i\left(\frac{2}{\Delta x}(x - x_j)\right)$, $\ell_i(s_k) = \delta_{i,k}$ and $s_k$ are the Gauss-Lobatto quadrature points on $[-1, 1]$

- Numerical integration on the same Gauss-Lobatto quadrature points:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} f(x)\,\mathrm{d}x \simeq \frac{\Delta x}{2}\sum_{k=0}^{p}\omega_k f(x_{k,j}) = \sum_{k=0}^{p}\omega_i f\left(x_j + \frac{\Delta x}{2}s_k\right)$$

[2] Florent Renac. "A robust high-order Lagrange-projection like scheme with large time steps for the isentropic Euler equations". In: *Numerische Mathematik* (2016), pp. 1–27. ISSN: 0945-3245. DOI: 10.1007/s00211-016-0807-0. URL: http://dx.doi.org/10.1007/s00211-016-0807-0.

# Time discretization

- Multiplication by a Lagrange polynomial and integration over a cell
- Development of the derivatives in time
- Numerical integration of the Lagrange polynomials
- Discretization of the derivatives in time

## Homogeneous relaxed Acoustic system without topography

$$\left\{\begin{aligned} \partial_t \tau - \partial_m u &= 0, \\ \partial_t u + \partial_m \pi &= 0, \\ \partial_t \pi + a^2 \partial_m u &= 0. \end{aligned}\right.$$

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## Homogeneous relaxed Acoustic system without topography

$$\left\{\begin{aligned} \int_{\kappa_j} \phi_{i,j}\,\partial_t \tau\,\mathrm{d}x - \int_{\kappa_j} \phi_{i,j}\,\partial_m u\,\mathrm{d}x &= 0, \\ \int_{\kappa_j} \phi_{i,j}\,\partial_t u\,\mathrm{d}x + \int_{\kappa_j} \phi_{i,j}\,\partial_m \pi\,\mathrm{d}x &= 0, \\ \int_{\kappa_j} \phi_{i,j}\,\partial_t \pi\,\mathrm{d}x + a^2 \int_{\kappa_j} \phi_{i,j}\,\partial_m u\,\mathrm{d}x &= 0. \end{aligned}\right.$$

# Time discretization

- Multiplication by a Lagrange polynomial and integration over a cell
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## Homogeneous relaxed Acoustic system without topography

$$\begin{cases} \sum_{k=0}^{p} \left( \int_{\kappa_j} \phi_{i,j}\phi_{k,j} \,\mathrm{dx} \right) \partial_t \tau_{k,j} - \int_{\kappa_j} \phi_{i,j}\, \partial_m u \,\mathrm{dx} = 0, \\ \sum_{k=0}^{p} \left( \int_{\kappa_j} \phi_{i,j}\phi_{k,j} \,\mathrm{dx} \right) \partial_t u_{k,j} + \int_{\kappa_j} \phi_{i,j}\, \partial_m \pi \,\mathrm{dx} = 0, \\ \sum_{k=0}^{p} \left( \int_{\kappa_j} \phi_{i,j}\phi_{k,j} \,\mathrm{dx} \right) \partial_t \pi_{k,j} + a^2 \int_{\kappa_j} \phi_{i,j}\, \partial_m u \,\mathrm{dx} = 0. \end{cases}$$

# Time discretization

- Multiplication by a Lagrange polynomial and integration over a cell
- Development of the derivatives in time
- Numerical integration of the Lagrange polynomials
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## Homogeneous relaxed Acoustic system without topography

$$\begin{cases} \dfrac{\Delta x}{2}\omega_i\,\partial_t\tau_{i,j} - \displaystyle\int_{\kappa_j}\phi_{i,j}\,\partial_m u\,\mathrm{d}x = 0, \\ \dfrac{\Delta x}{2}\omega_i\,\partial_t u_{i,j} + \displaystyle\int_{\kappa_j}\phi_{i,j}\,\partial_m \pi\,\mathrm{d}x = 0, \\ \dfrac{\Delta x}{2}\omega_i\,\partial_t\pi_{i,j} + a^2\displaystyle\int_{\kappa_j}\phi_{i,j}\,\partial_m u\,\mathrm{d}x = 0. \end{cases}$$

# Time discretization

- Multiplication by a Lagrange polynomial and integration over a cell
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## Homogeneous relaxed Acoustic system without topography

$$\begin{cases} \tau_{i,j}^{n+1^-} = \tau_{i,j}^n + \dfrac{2\Delta t}{\omega_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_m u \,\mathrm{d}x, \\ u_{i,j}^{n+1^-} = u_{i,j}^n - \dfrac{2\Delta t}{\omega_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_m \pi \,\mathrm{d}x, \\ \pi_{i,j}^{n+1^-} = \pi_{i,j}^n - a^2 \dfrac{2\Delta t}{\omega_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_m u \,\mathrm{d}x. \end{cases}$$

# Space discretization

- How to write equation on $\tau$ as in finite volume ?
- Approximation of the integral of $\partial_m u$
- Integration by part (exact)
- Introduction of the numerical fluxes
- Idem on $u$ and $\pi$

## Homogeneous relaxed Acoustic system without topography

$$\tau_{i,j}^{n+1^-} = \tau_{i,j}^{n} + \frac{2\Delta t}{\omega_i \Delta x} \int_{\kappa_j} \phi_{i,j}\, \partial_m u \,\mathrm{d}x$$

$$\updownarrow$$

$$\tau_j^{n+1^-} = \tau_j^n + \frac{\Delta t}{\Delta x} \tau_j^n \left( u_{j+1/2}^{*,n+1^-} - u_{j-1/2}^{*,n+1^-} \right) = L_j^{n+1^-} \tau_j^n$$

# Space discretization

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## Homogeneous relaxed Acoustic system without topography

$$\int_{\kappa_j} \phi_{i,j}\, \partial_m u \, \mathrm{d}x \simeq \frac{\Delta x}{2} \omega_i \tau_{i,j}^n \partial_x u_{i,j}^{n+1^-} = \tau_{i,j}^n \int_{\kappa_j} \phi_{i,j}\, \partial_x u \, \mathrm{d}x$$

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**Homogeneous relaxed Acoustic system without topography**

$$\int_{\kappa_j} \phi_{i,j}\, \partial_m u \,\mathrm{d}x \simeq \frac{\Delta x}{2} \omega_i \tau_{i,j}^n \partial_x u_{i,j}^{n+1^-} = \tau_{i,j}^n \int_{\kappa_j} \phi_{i,j}\, \partial_x u \,\mathrm{d}x$$

$$\simeq \tau_{i,j}^n \left( \int_{\kappa_j} \partial_x (\phi_{i,j} u) \,\mathrm{d}x - \int_{\kappa_j} u\, \partial_x \phi_{i,j} \,\mathrm{d}x \right)$$

# Space discretization

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## Homogeneous relaxed Acoustic system without topography

$$\int_{\kappa_j} \phi_{i,j}\,\partial_m u\,\mathrm{d}x \simeq \frac{\Delta x}{2}\omega_i \tau_{i,j}^n \partial_x u_{i,j}^{n+1^-} = \tau_{i,j}^n \int_{\kappa_j} \phi_{i,j}\,\partial_x u\,\mathrm{d}x$$

$$\simeq \tau_{i,j}^n \left( \int_{\kappa_j} \partial_x(\phi_{i,j} u)\,\mathrm{d}x - \int_{\kappa_j} u\,\partial_x \phi_{i,j}\,\mathrm{d}x \right)$$

$$\simeq \tau_{i,j}^n \left( \delta_{i,p} u_{j+1/2}^{*,n+1^-} - \delta_{i,0} u_{j-1/2}^{*,n+1^-} - \sum_{k=0}^{p} \omega_k u_{k,j}\,\partial_x \ell_i(s_k) \right)$$

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- Idem on $u$ and $\pi$

## Homogeneous relaxed Acoustic system without topography

$$\begin{cases} \tau_{i,j}^{n+1^-} = \tau_{i,j}^n + \dfrac{2\Delta t}{\omega_i \Delta x} \tau_{i,j}^n \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_x u \,\mathrm{d}x = L_{i,j}^{n+1^-} \tau_{i,j}^n, \\ u_{i,j}^{n+1^-} = u_{i,j}^n - \dfrac{2\Delta t}{\omega_i \Delta x} \tau_{i,j}^n \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_x \pi \,\mathrm{d}x, \\ \pi_{i,j}^{n+1^-} = \pi_{i,j}^n - a^2 \dfrac{2\Delta t}{\omega_i \Delta x} \tau_{i,j}^n \displaystyle\int_{\kappa_j} \phi_{i,j}\, \partial_x u \,\mathrm{d}x. \end{cases}$$

# Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X$
- Approximation of the integral of $X\,\partial_x u$ to bring out $L_{i,j}^{n+1^-}$
- Integration by part (not exact)
- Introduction of the numerical fluxes

## Homogeneous relaxed Acoustic system without topography

$$X_{i,j}^{n+1} = X_{i,j}^{n+1^-} - \frac{2\Delta t}{\omega_i \Delta x} \int_{\kappa_j} u\,\phi_{i,j}\,\partial_x X \,\mathrm{d}x$$

$$\updownarrow$$

$$X_j^{n+1} = L_j^{n+1^-} X_j^{n+1^-} - \frac{\Delta t}{\Delta x}\left((Xu)_{j+1/2}^{*,n+1^-} - (Xu)_{j-1/2}^{*,n+1^-}\right)$$

$$\text{with } (Xu)_{j+1/2}^{*} = \begin{cases} X_j\,u_{j+1/2}^{*}, & \text{if } u_{j+1/2}^{*} \geq 0, \\ X_{j+1}\,u_{j+1/2}^{*}, & \text{otherwise.} \end{cases}$$

# Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X$
- Approximation of the integral of $X\,\partial_x u$ to bring out $L_{i,j}^{n+1^-}$
- Integration by part (not exact)
- Introduction of the numerical fluxes

## Homogeneous relaxed Acoustic system without topography

$$\int_{\kappa_j} u\,\phi_{i,j}\,\partial_x X\,\mathrm{d}x = \int_{\kappa_j} \phi_{i,j}\,\partial_x(Xu)\,\mathrm{d}x - \int_{\kappa_j} X\,\phi_{i,j}\,\partial_x u\,\mathrm{d}x$$

# Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X$
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$$\simeq \int_{\kappa_j} \phi_{i,j}\,\partial_x(Xu)\,\mathrm{d}x - X_{i,j}^{n+1^-} \int_{\kappa_j} \phi_{i,j}\,\partial_x u\,\mathrm{d}x$$

# Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X$
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## Homogeneous relaxed Acoustic system without topography

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$$\simeq \int_{\kappa_j} \phi_{i,j}\,\partial_x(Xu)\,\mathrm{d}x - X_{i,j}^{n+1^-} \int_{\kappa_j} \phi_{i,j}\,\partial_x u\,\mathrm{d}x$$

$$\simeq \int_{\kappa_j} \partial_x(\phi_{i,j}Xu)\,\mathrm{d}x - \int_{\kappa_j} Xu\,\partial_x\phi_{i,j}\,\mathrm{d}x - X_{i,j}^{n+1^-} \int_{\kappa_j} \phi_{i,j}\,\partial_x u\,\mathrm{d}x$$

# Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
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$$\simeq \int_{\kappa_j} \partial_x(\phi_{i,j}Xu)\,\mathrm{d}x - \int_{\kappa_j} Xu\,\partial_x\phi_{i,j}\,\mathrm{d}x - X_{i,j}^{n+1^-} \int_{\kappa_j} \phi_{i,j}\,\partial_x u\,\mathrm{d}x$$

$$\longrightarrow \int_{\kappa_j} \partial_x(\phi_{i,j}Xu)\,\mathrm{d}x = \delta_{i,p}(Xu)_{j+1/2}^{*,n+1^-} - \delta_{i,0}(Xu)_{j-1/2}^{*,n+1^-}$$

# Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X$
- Approximation of the integral of $X\,\partial_x u$ to bring out $L_{i,j}^{n+1^-}$
- Integration by part (not exact)
- Introduction of the numerical fluxes

## Homogeneous relaxed Acoustic system without topography

$$\begin{cases} h_{i,j}^{n+1} = L_{i,j}^{n+1-} h_{i,j}^{n+1-} - \dfrac{2\Delta t}{w_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j} \partial_x (hu), \\ (hu)_{i,j}^{n+1} = L_{i,j}^{n+1-} (hu)_{i,j}^{n+1-} - \dfrac{2\Delta t}{w_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j} \partial_x (hu^2). \end{cases}$$

# Transport step

- How to write equation on $X = h, hu$ as in finite volume ?
- Rewriting the integral of $u\,\partial_x X$
- Approximation of the integral of $X\,\partial_x u$ to bring out $L_{i,j}^{n+1^-}$
- Integration by part (not exact)
- Introduction of the numerical fluxes

## Homogeneous relaxed Acoustic system without topography

$$\begin{cases} h_{i,j}^{n+1} = h_{i,j}^{n} - \dfrac{2\Delta t}{w_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j}\partial_x(hu), \\ (hu)_{i,j}^{n+1} = (hu)_{i,j}^{n} - \dfrac{2\Delta t}{w_i \Delta x} \displaystyle\int_{\kappa_j} \phi_{i,j}\partial_x(hu^2 + \pi). \end{cases}$$

# Contents

# Full explicit scheme

**Hypothesis :**

- $a > \max_j h_j \sqrt{g h_j}$
- $\max_j \frac{a \Delta t}{h_j \Delta x} \leq \frac{1}{2}$
- $\max_j \frac{\Delta t}{\Delta x} \left[ \left( u^*_{j-1/2} \right)_+ - \left( u^*_{j+1/2} \right)_- \right] \leq 1$

**Properties :**

- Conservative for $h$ (and for $hu$ if $z = \text{cst}$)
- $h_j^n > 0$, $\forall j, n$, provided that $h_j^0 > 0$, $\forall j$.
- Degeneration to classical Lagrange-Projection scheme if $z = \text{cst}$
- Well-balanced : preservation of the "lake at rest" conditions

# Implicit-explicit scheme

**Hypothesis :**

- $a > \max_j h_j \sqrt{g h_j}$
- $\max_j \frac{a\Delta t}{h_j \Delta x} \leq \frac{1}{2}$
- $\max_j \frac{\Delta t}{\Delta x} \left[ \left( u^*_{j-1/2} \right)_+ - \left( u^*_{j+1/2} \right)_- \right] \leq 1$

**Properties :**

- Similar properties as full explicit scheme
- **Prop :** It satisfies a discrete entropy inequality of the form :

$$\mathcal{U}_j^{n+1} - \mathcal{U}_j^n + \frac{\Delta t}{\Delta x_j} \left( \mathcal{F}_{j+1/2}^{n+1-} - \mathcal{F}_{j-1/2}^{n+1-} \right) \leq -\Delta t \, g \left\{ hu \partial_x z \right\}_j$$

where $\mathcal{U}$ and $\mathcal{F}$ are respectively a discretization of the total energy $hE = \frac{hu^2}{2} + \frac{gh^2}{2}$ and of the associated flux $(hE + p)\, u$.

# Implicit-explicit scheme

**Hypothesis :**

- $a > \max_j h_j \sqrt{g h_j}$
- $\max_j \frac{a \Delta t}{h_j \Delta x} \leq \frac{1}{2}$
- $\max_j \frac{\Delta t}{\Delta x} \left[ \left( u^*_{j-1/2} \right)_+ - \left( u^*_{j+1/2} \right)_- \right] \leq 1$

**Properties :**

- Similar properties as full explicit scheme
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where $\mathcal{U}$ and $\mathcal{F}$ are respectively a discretization of the total energy $hE = \frac{hu^2}{2} + \frac{gh^2}{2}$ and of the associated flux $(hE + p)\, u$.

**Ideas of proof :** Use of the characteristic variables $\overrightarrow{w} = \pi + au$ and $\overleftarrow{w} = \pi - au$ which only make sense in 1D. Jensen inequality and convexity of the total energy.

# Mean value in DG

$$\begin{cases} \overline{h}_j^{n+1} = \overline{h}_j^n - \dfrac{\Delta t}{\Delta x}\left((hu)^*_{j+1/2} - (hu)^*_{j-1/2}\right) \\ \overline{hu}_j^{n+1} = \overline{hu}_j^n - \dfrac{\Delta t}{\Delta x}\left((hu^2+\pi)^*_{j+1/2} - (hu^2+\pi)^*_{j-1/2}\right) \end{cases}$$

**Properties :**

- Conservative for $h$ and $hu$

# CFL condition for DG

**Hypothesis :**

- $a > \max_j max_i h_{i,j}\sqrt{gh_{i,j}}$
- $\frac{\Delta t}{\Delta x}\max_j \max_i c_{i,j} \leq 1$

with $c_{i,j} = \frac{2}{\omega_i}\left(\int u_j^{n+1-}(x)\partial_x\phi_{i,j}(x)dx - \delta_{i,p}u^*_{j+1/2,-} + \delta_{i,0}u^*_{j-1/2,+}\right)$

**Properties :**

- If $p = 0$ : $c_j = u^*_{j-1/2,+} - u^*_{j+1/2,-} \longrightarrow$ same CFL as in FV
- Convex combination :

$$\overline{X}_j^{n+1} = \sum_{i=0}^{p}\frac{\omega_i}{2}\left(1 - \frac{\Delta t}{\Delta x}c_{i,j}\right)X_{i,j}^{n+1-} + \frac{\Delta t}{\Delta x}(-u^*_{j+1/2,-})X_{0,j+1}^{n+1-} + \frac{\Delta t}{\Delta x}u^*_{j-1/2,+}X_{p,j-1}^{n+1-}$$

- $\overline{h}_j^{n+1} > 0$, $\forall j$, provided that $h_{i,j}^n > 0$, $\forall i, j$
- **Prop :** It satisfies a discrete entropy inequality of the form :

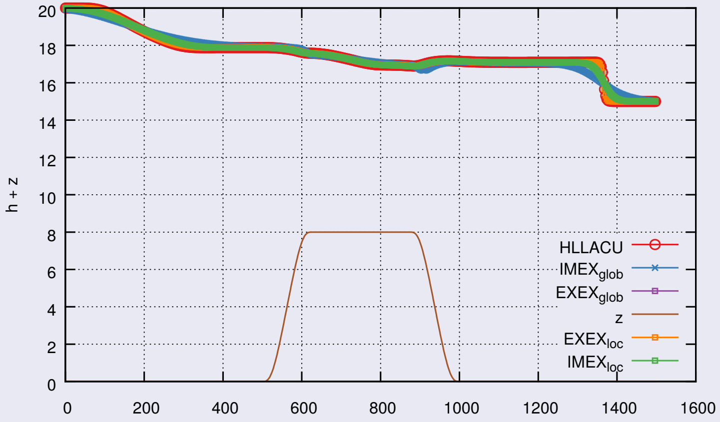
$$(\rho E)(\overline{\boldsymbol{U}}_j^{n+1}) - \overline{(\rho E)}_j^n + \frac{\Delta t}{\Delta x}\left[(\pi^*_{j+1/2} + (\rho E)^*_{j+1/2})u^*_{j+1/2} - (\pi^*_{j-1/2} + (\rho E)^*_{j-1/2})u^*_{j-1/2}\right] \leq 0.$$

# Contents

# Dam Break

Number of cells : 1500, final time : 50

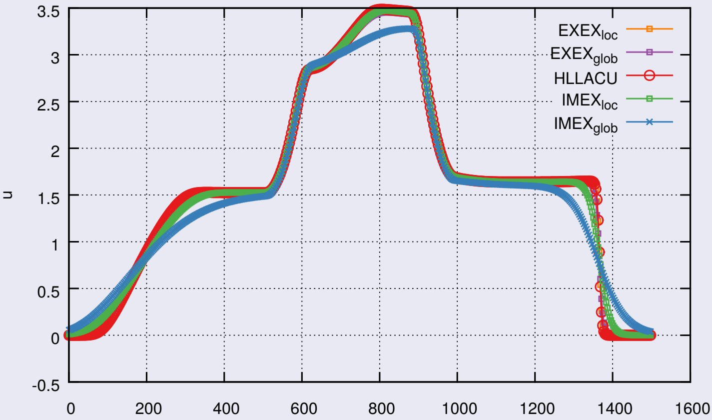
## Total height



# Dam Break

Number of cells : 1500, final time : 50

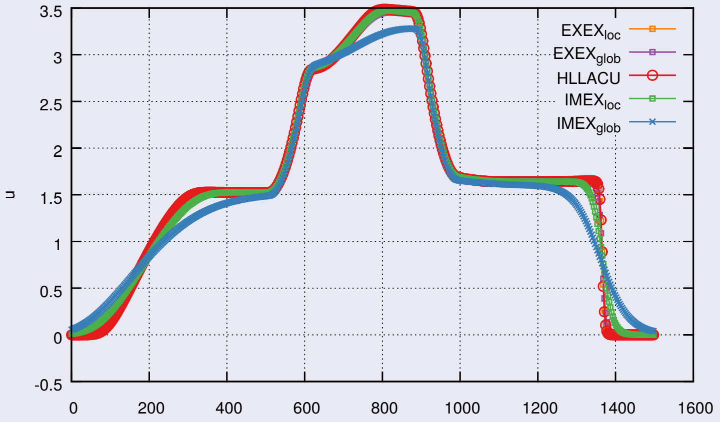
## Velocitiy



# Dam Break

Number of cells : 1500, final time : 50, 5× less time steps for IMEX

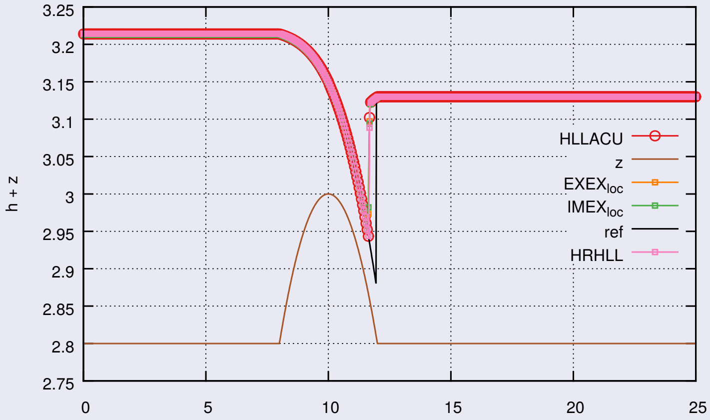
## Velocitiy



# Transcritical regime

Number of cells : 1600, final time : 200

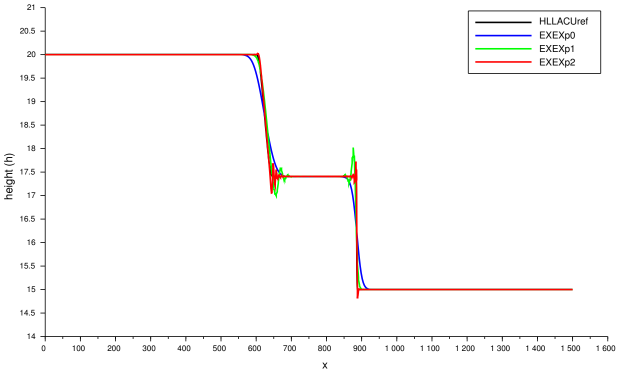
## Total height



# Dam break without topography

Number of cells : 500, final time : 15

## Total height



# Contents

# Conclusion

## Achievements

- New family of Lagrange-Projection type schemes for SWE
- Properties : consistent, Well-Balanced, entropy inequalities . . .
- Numerical results : diffusive as expected, but less restrictive CFL

## Perspectives

- Find stronger entropy inequalities (FV and DG)
- Add the source term for DG
- Study of low Froude flows for those schemes
- Show the numerical improvement in those regimes
- Multi-dimensional system
- Study other systems that have some asymptotic regime (eg. MHD)
- Use those schemes with AMR techniques in CanoP

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Thank you for your attention